ON THE DEGREE OF COCYCLES WITH VALUES IN THE GROUP $SU(2)$

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ABSTRACT

In this paper are presented some properties of smooth cocycles over irrational rotations on the circle with values in the group $SU(2)$. It is proved that the degree of any C^2-cocycle (the notion of degree was introduced in [2]) belongs to $2\pi\mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$). It is also shown that if the rotation satisfies a Diophantine condition, then every C^∞-cocycle with nonzero degree is C^∞-cohomologous to a cocycle of the form

$$\mathbb{T} \ni x \longmapsto \begin{bmatrix} e^{2\pi i(rx+w)} & 0 \\ 0 & e^{-2\pi i(rx+w)} \end{bmatrix} \in SU(2),$$

where $2\pi r$ is the degree of the cocycle and w is a real number. The above statement is false in the case of cocycles with zero degree. The proofs are based on ideas presented by R. Krikorian in [6].

1. Introduction

By $\mathbb{T}$ we will mean the circle group $\{z \in \mathbb{C}; |z| = 1\}$ which most often will be treated as the group $\mathbb{R}/\mathbb{Z}$; λ will denote Lebesgue measure on $\mathbb{T}$. For every $\gamma > 0$ we will identify functions on $\mathbb{R}/\gamma\mathbb{Z}$ with periodic of period γ functions on $\mathbb{R}$. Let $\alpha \in \mathbb{T}$ be an irrational number. We will denote by $T\colon (\mathbb{T}, \lambda) \to (\mathbb{T}, \lambda)$ the corresponding ergodic rotation $Tx = x + \alpha$.

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Let G be a compact Lie group, μ its Haar measure. Let $\varphi\colon \mathbb{T} \to G$ be a measurable function. Denote by $T_\varphi\colon (\mathbb{T}\times G, \lambda\otimes\mu) \to (\mathbb{T}\times G, \lambda\otimes\mu)$ the measure-preserving automorphism defined by

$$T_\varphi(x, g) = (Tx, g\cdot\varphi(x)),$$

called skew product. Every measurable function $\varphi\colon \mathbb{T} \to G$ determines the measurable cocycle over the rotation T given by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)\cdot\varphi(Tx)\cdot\ldots\cdot\varphi(T^{n-1}x) & \text{for } n > 0,\\ e & \text{for } n = 0,\\ (\varphi(T^n x)\cdot\varphi(T^{n+1}x)\cdot\ldots\cdot\varphi(T^{-1}x))^{-1} & \text{for } n < 0,\end{cases}$$

which we will identify with the function φ. Then $T^n_\varphi(x, g) = (Tx, g\cdot\varphi^{(n)}(x))$ for any integer n. Two cocycles $\varphi, \psi\colon \mathbb{T} \to G$ are **cohomologous** if there exists a measurable map $p\colon \mathbb{T} \to G$ such that

$$\varphi(x) = \psi_p(x) = p(x)^{-1}\psi(x)p(Tx).$$

For any $s \in \mathbb{N}\cup\{\infty\}$ if φ, ψ, p are of class C^s, then we will say that φ and ψ are **C^s-cohomologous**. If φ and ψ are cohomologous (C^s-cohomologous resp.), then the map $(x, g) \mapsto (x, p(x)g)$ establishes a metrical isomorphism (C^s-conjugation resp.) of T_φ and T_ψ.

In the case where G is the circle a lot of properties of a smooth cocycle $\varphi\colon \mathbb{T} \to \mathbb{T}$ and the associated skew product T_φ depend on the topological degree of φ denoted by $d(\varphi)$. For example, in [5] A. Iwanik, M. Lemańczyk and D. Rudolph have proved that if φ is a C^2-cocycle with $d(\varphi) \neq 0$, then T_φ is ergodic and it has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable. On the other hand, in [3] P. Gabriel, M. Lemańczyk and P. Liardet have proved that if φ is absolutely continuous with $d(\varphi) = 0$, then T_φ has singular spectrum. Moreover, if α is Diophantine, then every C^∞-cocycle $\varphi\colon \mathbb{T} \to \mathbb{T}$ is C^∞-cohomologous to a cocycle of the form $\mathbb{T} \ni x \mapsto e^{2\pi i(d(\varphi)x+w)} \in \mathbb{T}$, where w is a real number.

The aim of this paper is to study how the value of degree influences properties of cocycles in the case where $G = SU(2)$ (the notion of degree for cocycles with values in $SU(2)$ was introduced in [2]).

1.1 NOTATION. For a given matrix $A = [a_{ij}]_{i,j=1,2} \in M_2(\mathbb{C})$ define the norm of A by $\|A\| = \sqrt{\frac{1}{2}\sum_{i,j=1}^{2}|a_{ij}|^2}$. Observe that if A is an element of the Lie algebra $\mathfrak{su}(2)$, i.e.,

$$A = \begin{bmatrix} ia & b+ic \\ -b+ic & -ia \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$, then $\|A\| = \sqrt{\det A}$. Moreover, if B is an element of the group $SU(2)$, i.e.,

$$A = \begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}$, $|z_1|^2+|z_2|^2 = 1$, then $\operatorname{Ad}_B A = BAB^{-1} \in \mathfrak{su}(2)$ and $\|\operatorname{Ad}_B A\| = \|A\|$. By $\mathfrak{T}$ we will mean the maximal torus in $SU(2)$, i.e., the subgroup of $SU(2)$ containing all matrices of the form $\begin{bmatrix} z & 0 \\ 0 & \overline{z} \end{bmatrix}$, where $z \in \mathbb{T}$.

Let X be a Riemann manifold. Assume that X is compact. Then by $L^1(X, \mathfrak{su}(2))$ we mean the space of all functions $f\colon X \to \mathfrak{su}(2)$ such that

$$\|f\|_{L^1(X)} = \int_X \|f(x)\| d\lambda(x) < \infty,$$

where λ is normalized Lebesgue measure on X. The space $L^1(X, \mathfrak{su}(2))$ endowed with the norm $\|\cdot\|_{L^1}$ is a Banach space. Consider the scalar product of $\mathfrak{su}(2)$ given by

$$\langle A, B\rangle = -\frac{1}{8}\operatorname{tr}(\operatorname{Ad} A \circ \operatorname{Ad} B).$$

Then $\|A\| = \sqrt{\langle A, A\rangle}$. By $L^2(X, \mathfrak{su}(2))$ we mean the space of all functions $f\colon X \to \mathfrak{su}(2)$ such that

$$\|f\|_{L^2(X)} = \sqrt{\int_X \|f(x)\|^2 d\lambda(x)} < \infty.$$

For two $f_1, f_2 \in L^2(X, \mathfrak{su}(2))$ set

$$\langle f_1, f_2\rangle_{L^2(X)} = \int_X \langle f_1(x), f_2(x)\rangle d\lambda(x).$$

The space $L^2(X, \mathfrak{su}(2))$ endowed with the above scalar product is a Hilbert space.

For every $s \in \mathbb{N}\cup\{\infty\}$ we will denote by $C^s(X, SU(2))$ the set all C^s-functions on X with values in $SU(2)$. For any $\varphi \in C^1(\mathbb{R}, SU(2))$ denote by $L(\varphi)\colon \mathbb{R} \to \mathfrak{su}(2)$ the function $L(\varphi)(x) = D\varphi(x)(\varphi(x))^{-1}$. For any C^s-function $\psi\colon \mathbb{T} \to \mathfrak{su}(2)$ ($s \in \mathbb{N}$) set

$$\|\psi\|_{C^s} = \max_{0\le k\le s} \sup_{x\in\mathbb{T}} \|D^k\psi(x)\|.$$

For two $\varphi, \psi \in C^s(\mathbb{T}, SU(2))$ define

$$\|\varphi - \psi\|_{C^s} = \max(\sup_{x\in\mathbb{T}} \|\varphi(x) - \psi(x)\|, \|L(\varphi) - L(\psi)\|_{C^{s-1}}).$$

Then $(C^s(\mathbb{T}, SU(2)), \|\cdot - \cdot\|_{C^s})$ is a metric space for any natural s.

1.2 Definition of degree and basic properties. One definition of degree for cocycles with values in $SU(2)$ was given in [2]. The following result establishes the base of this definition.

Theorem 1.1: *For every $\varphi \in C^1(\mathbb{T}, SU(2))$ there exists a measurable and bounded function $\psi: \mathbb{T} \to \mathfrak{su}(2)$ such that*

$$\frac{1}{n} L(\varphi^{(n)}) \to \psi \text{ in } L^1(\mathbb{T}, \mathfrak{su}(2)) \text{ and } \lambda\text{-almost everywhere},$$

as $n \to \pm\infty$. Moreover, $\operatorname{Ad}_{\varphi(x)} \psi(Tx) = \psi(x)$ and $\|\psi(x)\|$ is constant for a.e. $x \in \mathbb{T}$.

Proof: First notice that

$$L(\varphi^{(n)}) = \sum_{k=0}^{n-1} \operatorname{Ad}_{\varphi^{(k)}}(L(\varphi) \circ T^k),$$

$$L(\varphi^{(-n)}) = -\sum_{k=1}^{n} \operatorname{Ad}_{\varphi^{(-k)}}(L(\varphi) \circ T^{-k}) = -\operatorname{Ad}_{\varphi^{(-n)}}(L(\varphi^{(n)}) \circ T^{-n}),$$

for any natural n. Let us consider the unitary operator

$$U: L^2(\mathbb{T}, \mathfrak{su}(2)) \to L^2(\mathbb{T}, \mathfrak{su}(2)), \quad Uf(x) = \operatorname{Ad}_{\varphi(x)} f(Tx).$$

Then $U^n f(x) = \operatorname{Ad}_{\varphi^{(n)}(x)} f(T^n x)$ and $U^{-n} f(x) = \operatorname{Ad}_{\varphi^{(-n)}(x)} f(T^{-n} x)$ for any natural n. Therefore

$$\frac{1}{n} L(\varphi^{(n)}) = \frac{1}{n} \sum_{k=0}^{n-1} U^k(L(\varphi)) \quad \text{and} \quad \frac{1}{n} L(\varphi^{(n)}) = -\frac{1}{n} \sum_{k=1}^{n} U^{-k}(L(\varphi)).$$

By the von Neuman ergodic theorem, there exist U-invariant $\psi_+, \psi_- \in L^2(\mathbb{T}, \mathfrak{su}(2))$ such that

$$\lim_{n\to+\infty} \frac{1}{n} L(\varphi^{(n)}) = \psi_+ \quad \text{and} \quad \lim_{n\to+\infty} \frac{1}{n} L(\varphi^{(-n)}) = \psi_-$$

in $L^2(\mathbb{T}, \mathfrak{su}(2))$. Next observe that

$$\begin{aligned}\left\|\frac{1}{n} L(\varphi^{(-n)})(x) + \psi_+(x)\right\| &= \left\| \operatorname{Ad}_{\varphi^{(-n)}} \left(-\frac{1}{n} L(\varphi^{(n)})(T^{-n}x) + \psi_+(T^{-n}x)\right)\right\| \\ &= \left\|\frac{1}{n} L(\varphi^{(n)})(T^{-n}x) - \psi_+(T^{-n}x)\right\|.\end{aligned}$$

It follows that $\psi = \psi_+ = -\psi_-$. Moreover, $\|\psi(x)\| = \|\operatorname{Ad}_{\varphi(x)} \psi(Tx)\| = \|\psi(Tx)\|$. Hence $\|\psi(x)\|$ is constant for a.e. $x \in \mathbb{T}$, by the ergodicity of T.

Let $\tilde{\varphi} \in L^2(\mathbb{T} \times SU(2), \mathfrak{su}(2))$ be given by $\tilde{\varphi}(x, g) = \mathrm{Ad}_g\, L(\varphi)(x)$. Then

$$\tilde{\varphi}(T^n_\varphi(x, g)) = \mathrm{Ad}_g(U^n(L(\varphi))(x))$$

for any integer n. Therefore

$$\mathrm{Ad}_g\left(\frac{1}{n}L(\varphi^{(n)})(x)\right) = \frac{1}{n}\sum_{k=0}^{n-1}\tilde{\varphi}(T^k_\varphi(x, g))$$

and

$$\mathrm{Ad}_g\left(\frac{1}{n}L(\varphi^{(-n)})(x)\right) = \frac{1}{n}\sum_{k=1}^{n}\tilde{\varphi}(T^{-k}_\varphi(x, g)).$$

By the Birkhoff ergodic theorem $\frac{1}{n}\sum_{k=0}^{n-1}\tilde{\varphi}(T^k_\varphi(x, g))$ and $\frac{1}{n}\sum_{k=1}^{n}\tilde{\varphi}(T^{-k}_\varphi(x, g))$ converge for $\lambda \otimes \mu$-a.e. $(x, g) \in \mathbb{T} \times SU(2)$ (μ is the normalized Haar measure of $SU(2)$). Consequently, by the Fubini theorem, $\frac{1}{n}L(\varphi^{(n)})(x)$ and $\frac{1}{n}L(\varphi^{(-n)})(x)$ converge for a.e. $x \in \mathbb{T}$, which completes the proof. ∎

Definition 1: The number

$$\|\varphi\| = \lim_{n\to\pm\infty}\left\|\frac{1}{n}L(\varphi^{(n)})\right\|_{L^1(\mathbb{T})} = \inf_{n\in\mathbb{Z}\setminus\{0\}}\left\|\frac{1}{n}L(\varphi^{(n)})\right\|_{L^1(\mathbb{T})}$$

will be called the **degree** of the cocycle φ and denoted by $d(\varphi)$.

It is easy to check that degree is invariant under the relation of C^1-cohomology. The following theorem indicates an important property of cocycles with nonzero degree.

THEOREM 1.2 (see [2]): *Suppose that $\varphi\colon \mathbb{T} \to SU(2)$ is a C^1-cocycle with $d(\varphi) \neq 0$. Then the skew product is not ergodic and φ is cohomologous to a measurable cocycle of the form*

$$\mathbb{T} \ni x \mapsto \begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix} \in \mathfrak{T},$$

where $\gamma\colon \mathbb{T} \to \mathbb{T}$ is a measurable function. Moreover, all ergodic components of T_φ are metrically isomorphic to the skew product $T_\gamma\colon \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ and T_γ is mixing on the orthocomplement of the space of functions depending only on the first variable.

1.3 MAIN RESULTS. An important question is: what can one say on values of degree? It is easy to see that if a cocycle φ is cohomologous to a cocycle with values in the subgroup $\mathfrak{T}$ via a smooth transfer function, then $d(\varphi) \in 2\pi\mathbb{N}$. Moreover, if α is the golden ratio, then the degree of any C^2-cocycle belongs to

$2\pi\mathbb{N}$ (see [2]). In the paper we extend this result to all irrational α (see Theorem 2.7). A completely different phenomenon occurs in the case of cocycles over multidimensional rotations. For details we refer to [2, §8,9]. Moreover, we prove that degree is invariant under the relation of measurable cohomology (see Theorem 2.10). The proofs of Theorem 2.7 and 2.10 are based on the renormalization algorithm for some $\mathbb{Z}^2$-actions on $\mathbb{R}\times SU(2)$ presented by R. Krikorian in [6] and on the following result.

THEOREM 1.3: *For every C^2-cocycle $\varphi\colon \mathbb{T}\to SU(2)$ we have*

$$\lim_{n\to\pm\infty}\frac{1}{n^2}DL(\varphi^{(n)})(x)=0$$

for a.e. $x\in\mathbb{T}$.

Proof: First observe that

$$\frac{1}{n^2}DL(\varphi^{(n)})=\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}[U^jL(\varphi),U^kL(\varphi)]+\frac{1}{n^2}\sum_{k=0}^{n-1}U^k(DL(\varphi)),$$

$$\frac{1}{n^2}DL(\varphi^{(-n)})=-\frac{1}{n^2}\sum_{k=1}^{n}\sum_{j=1}^{k-1}[U^{-j}L(\varphi),U^{-k}L(\varphi)]-\frac{1}{n^2}\sum_{k=1}^{n}U^{-k}(DL(\varphi)),$$

for any natural $n\neq 0$. Next note that if $\{a_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathfrak{su}(2)$ such that $\frac{1}{n}\sum_{k=1}^{n}a_k$ converges, as $n\to+\infty$, then

$$\frac{1}{n^2}\sum_{k=1}^{n}\sum_{j=1}^{k-1}[a_j,a_k]$$

tends to zero, as $n\to+\infty$. This follows by the same method as in [2, Prop. 6.6]. It follows that

$$\lim_{n\to\pm\infty}\frac{1}{n^2}DL(\varphi^{(n)})(x)=0$$

whenever

$$\lim_{n\to\pm\infty}\frac{1}{n}L(\varphi^{(n)})(x)$$

exists. Now applying Theorem 1.1 completes the proof. ∎

Now assuming Theorem 2.7 we get the following simple conclusion.

COROLLARY 1.4: *Let α be an irrational number and let $A\colon \mathbb{T}\to SU(2)$ be a constant cocycle. Suppose that $\varphi\colon \mathbb{T}\to SU(2)$ is a C^2-cocycle such that $\|\varphi-A\|_{C^1}<2\pi$. Then $d(\varphi)=0$.*

Proof: Since $d(\varphi)\leq\|L(\varphi)\|_{C^0}$ and $\|L(\varphi)\|_{C^0}<2\pi$, we have $d(\varphi)<2\pi$. As $d(\varphi)\in 2\pi\mathbb{N}$ we obtain $d(\varphi)=0$. ∎

We should compare this with the following result, which is due to M. Herman [4].

PROPOSITION 1.5: *For any irrational α the closure (in the C^∞-topology) of the set of all C^∞-cocycles which are not C^∞-cohomologous to any constant cocycle contains the set of all constant cocycles.*

It follows that for any irrational α there exists a C^∞-cocycle $\varphi\colon \mathbb{T} \to SU(2)$ with $d(\varphi) = 0$ which is not C^∞-cohomologous to any constant cocycle. For any $r \in \mathbb{N}$ and $w \in \mathbb{R}$ we will denote by $\exp_{r,w}\colon \mathbb{T} \to SU(2)$ the cocycle $\exp_{r,w}(x) = e^{2\pi(rx+w)h}$, where $h = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. On the other hand, in Section 3 (see Theorem 3.1) we show that if α satisfies a Diophantine condition, then every C^∞-cocycle $\varphi\colon \mathbb{T} \to SU(2)$ with $d(\varphi) = 2\pi r \neq 0$ is C^∞-cohomologous to a cocycle $\exp_{r,w}$ (this result has been independently observed by R. Krikorian but has not been published). This indicates the next essential difference between cocycles with zero and nonzero degree. The proof of Theorem 3.1 is based on a result (see Proposition 3.3) describing C^∞-cocycles in some neighborhood of the cocycle $\exp_{r,0}$, and which was proved by R. Krikorian [6, Th. 9.1].

2. Values of degree

2.1 $\mathbb{Z}^2$-ACTIONS ON $\mathbb{R} \times SU(2)$ AND THE RENORMALIZATION ALGORITHM. For the background of the contents of this section we refer the reader to [6]. Let $s \in \mathbb{N} \cup \{\infty\}$. For any $\alpha \in \mathbb{R}$ and $A \in C^s(\mathbb{R}, SU(2))$ we will denote by $(\alpha, A) : \mathbb{R} \times SU(2) \to \mathbb{R} \times SU(2)$ the skew product

$$(\alpha, A)(x, g) = (x + \alpha, g \cdot A(x)).$$

Let α be an irrational number. We will consider $\mathbb{Z}^2$-actions on $\mathbb{R} \times SU(2)$ of the form $((1, C), (\alpha, A))$, where $A, C \in C^s(\mathbb{R}, SU(2))$, i.e., $\mathbb{Z}^2$-actions generated by commuting skew products $(1, C)$ and (α, A). Suppose that $((1, C), (\alpha, A))$ is a $\mathbb{Z}^2$-action. Then

$$A(x) \cdot C(x + \alpha) = C(x) \cdot A(x + 1)$$

for any real x. Note also that if $C \equiv \mathrm{Id}$, then $A\colon \mathbb{R} \to SU(2)$ is a periodic function of period 1. Therefore we can identify any cocycle $A\colon \mathbb{T} \to SU(2)$ over the rotation $Tx = x + \alpha$ with a $\mathbb{Z}^2$-action $((1, \mathrm{Id}), (\alpha, A))$. We can extend also the relation of cohomology to $\mathbb{Z}^2$-actions. Two $\mathbb{Z}^2$-actions $((1, C_1), (\alpha, A_1))$ and $((1, C_2), (\alpha, A_2))$ are C^s-**cohomologous** if there exists $B \in C^s(\mathbb{R}, SU(2))$ such that

$$(0,B)\circ(1,C_1)\circ(0,B)^{-1}=(1,C_2),$$
$$(0,B)\circ(\alpha,A_1)\circ(0,B)^{-1}=(\alpha,A_2),$$

or equivalently

$$B(x)^{-1}\cdot C_1(x)\cdot B(x+1)=C_2(x),$$
$$B(x)^{-1}\cdot A_1(x)\cdot B(x+\alpha)=A_2(x).$$

Notice that every $\mathbb{Z}^2$-action $((1,C),(\alpha,A))$, where $A,C\in C^s(\mathbb{R},SU(2))$, is C^s-cohomologous to a cocycle $((1,\mathrm{Id}),(\alpha,\tilde{A}))$. For details we refer to [6].

Assume that $\alpha\in[0,1)$ is an irrational number with continued fraction expansion

$$\alpha=[0;a_1,a_2,\ldots].$$

Let $(p_k/q_k)_{k=-1}^{\infty}$ be the convergents of α ($p_{-1}=1$, $q_{-1}=0$). For every $k\geq -1$ set

$$\beta_k=(-1)^k(q_k\alpha-p_k)\quad\text{and}\quad\alpha_k=[0;a_{k+1},a_{k+2},\ldots].$$

Then

$$\frac{1}{q_k+q_{k+1}}<\beta_k<\frac{1}{q_{k+1}},\tag{1}$$
$$\beta_{k-2}=a_k\beta_{k-1}+\beta_k,\tag{2}$$
$$\beta_k=\alpha_0\cdot\alpha_1\cdot\ldots\cdot\alpha_k,\tag{3}$$
$$\beta_kq_{k+1}+\beta_{k+1}q_k=1.\tag{4}$$

Let $((1,C),(\alpha,A))$ be a $\mathbb{Z}^2$-action such that $A,C\in C^s(\mathbb{R},SU(2))$. Consider the sequence $\{(U_k,V_k)\}_{k=0}^{\infty}$ of $\mathbb{Z}^2$-actions defined by

$$(U_0,V_0)=((1,C),(\alpha,A)),$$
$$(U_k,V_k)=(V_{k-1},V_{k-1}^{-a_k}U_{k-1})\quad\text{for } k\geq 1.$$

Set $\mathcal{R}_k((1,C),(\alpha,A))=(U_k,V_k)$. Then

$$\mathcal{R}_k((1,C),(\alpha,A))=((U_0^{-p_{k-1}}V_0^{q_{k-1}})^{(-1)^{k-1}},(U_0^{-p_k}V_0^{q_k})^{(-1)^k})$$
$$=((\beta_{k-1},C_k),(\beta_k,A_k)),$$

where $A_k,C_k\in C^s(\mathbb{R},SU(2))$. Note that

$$(U_0,V_0)=(U_k^{q_k}V_k^{q_{k-1}},U_k^{p_k}V_k^{p_{k-1}}).\tag{5}$$

Observe that if $C\equiv\mathrm{Id}$, then

$$C_k=A^{((-1)^{k-1}q_{k-1})}\quad\text{and}\quad A_k=A^{((-1)^kq_k)}.\tag{6}$$

For every $k \in \mathbb{N}$ set

$$\tilde{C}_k(x) = C_k(\beta_{k-1}x) \quad \text{and} \quad \tilde{A}_k(x) = A_k(\beta_{k-1}x).$$

We will also consider the renormalizations of $((1, C), (\alpha, A))$ defined by

$$\widetilde{\mathcal{R}}_k((1, C), (\alpha, A)) = ((1, \tilde{C}_k), (\alpha_k, \tilde{A}_k)).$$

2.2 DEGREE OF $\mathbb{Z}^2$-ACTIONS. Suppose that $A, C \in C^1(\mathbb{R}/\gamma\mathbb{Z}, SU(2))$, where $\gamma > 0$. Then $A_k, C_k \in C^1(\mathbb{R}/\gamma\mathbb{Z}, SU(2))$ for any $k \in \mathbb{N}$. Define

$$d_k = d_k((1, C), (\alpha, A)) = \beta_k \|L(C_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} + \beta_{k-1}\|L(A_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})}.$$

Of course, d_k does not depend on the choice of γ, because we always consider normalized Lebesgue measure on $\mathbb{R}/\gamma\mathbb{Z}$. Observe that $d_k \leq d_{k-1}$. Indeed, since

$$\begin{aligned} A_k(x) &= C_{k-1}(x)A_{k-1}(x + \beta_{k-2} - \beta_{k-1}) \ldots A_{k-1}(x + \beta_{k-2} - a_k\beta_{k-1}), \\ C_k(x) &= A_{k-1}(x), \end{aligned}$$

we have

$$\begin{aligned} \|L(A_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} \leq& \|L(C_{k-1})\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} + a_k\|L(A_{k-1})\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})}, \\ \|L(C_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} =& \|L(A_{k-1})\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})}. \end{aligned}$$

It follows that

$$\begin{aligned} d_k &= \beta_k\|L(C_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} + \beta_{k-1}\|L(A_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} \\ &\leq \beta_{k-1}\|L(C_{k-1})\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} + (a_k\beta_{k-1} + \beta_k)\|L(A_{k-1})\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} \\ &= d_{k-1}, \end{aligned}$$

by (2).

Definition 2: The number

$$d((1, C), (\alpha, A)) = \lim_{k\to\infty} d_k((1, C), (\alpha, A))$$

will be called the **degree** of the $\mathbb{Z}^2$-action $((1, C), (\alpha, A))$.

Of course, we should check that the above definition is the extension of Definition 1. Suppose that $A = \varphi \in C^1(\mathbb{T}, SU(2))$ and $C \equiv \mathrm{Id}$. By Definition 1 and (6),

$$\lim_{k\to\infty} \frac{1}{q_k}\|L(A_k)\|_{L^1(\mathbb{T})} = \lim_{k\to\infty} \frac{1}{q_{k-1}}\|L(C_k)\|_{L^1(\mathbb{T})} = d(\varphi).$$

Since $\beta_k q_{k+1} + \beta_{k+1}q_k = 1$, we obtain $d(\varphi) = d((1, C), (\alpha, A))$.

In the following two lemmas are presented fundamental properties of degree.

LEMMA 2.1: *Let $A, C \in C^1(\mathbb{R}/\gamma\mathbb{Z}, SU(2))$ and $\varphi \in C^1(\mathbb{T}, SU(2))$. Suppose that the $\mathbb{Z}^2$-actions $((1,C),(\alpha,A))$ and $((1,\mathrm{Id}),(\alpha,\varphi))$ are C^1-cohomologous. Then $d(\varphi) = d((1,C),(\alpha,A))$.*

Proof: Let $B: \mathbb{R} \to SU(2)$ be a C^1-function such that

$$(0,B)\circ(1,C)\circ(0,B)^{-1} = (1,\mathrm{Id}),$$
$$(0,B)\circ(\alpha,A)\circ(0,B)^{-1} = (\alpha,\varphi).$$

Then

$$(0,B)\circ(\beta_{k-1},C_k)\circ(0,B)^{-1} = (\beta_{k-1},\varphi^{((-1)^{k-1}q_{k-1})}),$$
$$(0,B)\circ(\beta_k,A_k)\circ(0,B)^{-1} = (\beta_k,\varphi^{((-1)^k q_k)}).$$

Hence

$$B(x)^{-1}\cdot C_k(x)\cdot B(x+\beta_{k-1}) = \varphi^{((-1)^{k-1}q_{k-1})}(x),$$
$$B(x)^{-1}\cdot A_k(x)\cdot B(x+\beta_k) = \varphi^{((-1)^k q_k)}(x).$$

It follows that

$$\left|\|L(C_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} - \|L(\varphi^{((-1)^{k-1}q_{k-1})})\|_{L^1([0,\gamma))}\right| \leq \|L(B)\|_{L^1([0,\gamma))} + \|L(B)\|_{L^1([\beta_{k-1},\beta_{k-1}+\gamma))}$$

and

$$\left|\|L(A_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} - \|L(\varphi^{((-1)^k q_k)})\|_{L^1([0,\gamma))}\right| \leq \|L(B)\|_{L^1([0,\gamma))} + \|L(B)\|_{L^1([\beta_k,\beta_k+\gamma))}.$$

Since $\frac{1}{n}\|L(\varphi^{(n)})(x)\| \to d(\varphi)$ in $L^1(\mathbb{T},\mathbb{R})$, as $n \to \pm\infty$ (by Theorem 1.1), we have

$$\frac{1}{q_k}\|L(\varphi^{((-1)^k q_k)})\|_{L^1([0,\gamma))} \to d(\varphi).$$

Moreover, $\|L(B)\|_{L^1([\beta_k,\beta_k+\gamma))} \leq 2\|L(B)\|_{L^1([0,2\gamma))}$. Therefore

$$\lim_{k\to\infty}\frac{1}{q_k}\|L(A_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} = \lim_{k\to\infty}\frac{1}{q_{k-1}}\|L(C_k)\|_{L^1(\mathbb{R}/\gamma\mathbb{Z})} = d(\varphi).$$

Since $\beta_k q_{k+1} + \beta_{k+1}q_k = 1$, we obtain $d(\varphi) = d((1,C),(\alpha,A))$. ∎

Let $\varphi \in C^1(\mathbb{T}, SU(2))$ and let $((1,\tilde{C}_k),(\alpha_k,\tilde{A}_k)) = \widetilde{\mathcal{R}}_k((1,\mathrm{Id}),(\alpha,\varphi))$. Then $\tilde{A}_k, \tilde{C}_k \in C^1(\mathbb{R}/\beta_{k-1}^{-1}\mathbb{Z}, SU(2))$.

LEMMA 2.2: $d((1,\tilde{C}_k),(\alpha_k,\tilde{A}_k))=d(\varphi)$.

Proof: For every $\gamma>0$, by $S_\gamma\colon \mathbb{R}\to\mathbb{R}$ we mean the linear scaling $S_\gamma x=\gamma x$. It is easy to check that

$$\mathcal{R}_n((1,\tilde{C}_k),(\alpha_k,\tilde{A}_k)) = ((\beta_{n+k-1}/\beta_{k-1}, C_{n+k}\circ S_{\beta_{k-1}}),(\beta_{n+k}/\beta_{k-1}, A_{n+k}\circ S_{\beta_{k-1}})).$$

Therefore

$$\begin{aligned} d_n((1,\tilde{C}_k),(\alpha_k,\tilde{A}_k)) =&\beta_{n+k-1}/\beta_{k-1}\|L(C_{n+k}\circ S_{\beta_{k-1}})\|_{L^1([0,\beta_{k-1}^{-1}))}\\ &+\beta_{n+k}/\beta_{k-1}\|L(A_{n+k}\circ S_{\beta_{k-1}})\|_{L^1([0,\beta_{k-1}^{-1}))}\\ =&\beta_{n+k-1}\int_0^{\beta_{k-1}^{-1}}\|L(C_{n+k}\circ S_{\beta_{k-1}})(x)\|dx\\ &+\beta_{n+k}\int_0^{\beta_{k-1}^{-1}}\|L(A_{n+k}\circ S_{\beta_{k-1}})(x)\|dx\\ =&\beta_{n+k-1}\|L(C_{n+k})\|_{L^1(\mathbb{T})}+\beta_{n+k}\|L(A_{n+k})\|_{L^1(\mathbb{T})}. \end{aligned}$$

It follows that $d_n((1,\tilde{C}_k),(\alpha_k,\tilde{A}_k))\to d(\varphi)$, as $n\to\infty$, which proves the lemma. ∎

We recall a quantity $J(\varphi)$ introduced in [6]. For any function $\varphi\colon\mathbb{R}\to SU(2)$ and $y\in\mathbb{T}$ we will denote by $\varphi_y\colon\mathbb{R}\to SU(2)$ the function $\varphi_y(x)=\varphi(x+y)$. Write $((1,\tilde{C}_{k,y}),(\alpha_k,\tilde{A}_{k,y}))=\widetilde{\mathcal{R}}_k((1,\mathrm{Id}),(\alpha,\varphi_y))$. Then

$$\tilde{C}_{k,y}(x)=\tilde{C}_k(x+\beta_{k-1}^{-1}y)\quad\text{and}\quad \tilde{A}_{k,y}(x)=\tilde{A}_k(x+\beta_{k-1}^{-1}y).$$

Let $\varphi\colon\mathbb{T}\to SU(2)$ be a C^1-cocycle. For every $y\in\mathbb{T}$ define

$$\begin{aligned} J_k(y)&=\int_0^1\|L(\tilde{A}_{k,y})(x)\|dx+\int_0^{\alpha_k}\|L(\tilde{C}_{k,y})(x)\|dx\\ &=\int_y^{y+\beta_{k-1}}\|L(A_k)(x)\|dx+\int_y^{y+\beta_k}\|L(C_k)(x)\|dx. \end{aligned}$$

It is easy to check (see [6]) that $J_k(y)\le J_{k-1}(y)$. Let $J\colon\mathbb{T}\to\mathbb{R}$ be given by $J(y)=\lim_{k\to\infty}J_k(y)$. Next note that $J(y+\alpha)=J(y)$ for any $y\in\mathbb{T}$. Indeed, first observe that

$$\varphi^{(n)}(x-\alpha)=\varphi(x-\alpha)\cdot\varphi^{(n)}(x)\cdot\varphi(x+(n-1)\alpha)^{-1}$$

for any integer n. Hence

$$\big|\|L(\varphi^{(n)})(x-\alpha)\|-\|L(\varphi^{(n)})(x)\|\big|\le\|L(\varphi(x-\alpha)\|+\|L(\varphi(x+(n-1)\alpha)\|.$$

It follows that for every $y \in \mathbb{T}$ and $i = -1, 2$ we have

$$\left| \int_{y+\alpha}^{y+\alpha+\beta_{k+i}} \|L(\varphi^{((-1)^k q_k)})(x)\| dx - \int_{y}^{y+\beta_{k+i}} \|L(\varphi^{((-1)^k q_k)})(x)\| dx \right|$$
$$\leq \int_{y}^{y+\beta_{k+i}} \|L(\varphi)(x)\| dx + \int_{y+(-1)^k\beta_k}^{y+(-1)^k\beta_k+\beta_{k+i}} \|L(\varphi)(x)\| dx \to 0,$$

because $\beta_{k+i} \to 0$. Therefore $J(y+\alpha) = J(y)$, by (6). Since $J: \mathbb{T} \to \mathbb{R}$ is the limit (the convergence is pointwise) of a decreasing sequence of continuous functions, it follows that J is constant. Define $J(\varphi) = J(y)$ for any $y \in \mathbb{T}$. In the next section we show that if $\varphi \in C^2(\mathbb{T}, SU(2))$, then $J(\varphi) = d(\varphi)$.

2.3 Fundamental lemmas and the first main theorem. Suppose that $\varphi: \mathbb{T} \to SU(2)$ is a C^2-cocycle. Let $\psi: \mathbb{T} \to \mathsf{su}(2)$ be a measurable function such that

$$\frac{1}{n} L(\varphi^{(n)}) \to \psi \text{ in } L^1(\mathbb{T}, \mathsf{su}(2)) \text{ and almost everywhere,}$$

as $n \to \pm\infty$, $\mathrm{Ad}_{\varphi^{(m)}}(\psi \circ T^m) = \psi$ and $\|\psi(x)\| = d(\varphi)$ for a.e. $x \in \mathbb{T}$ (see Theorem 1.1).

We give a few asymptotic properties of the renormalization $\widetilde{\mathcal{R}}$ which we will need in proofs of the main theorems.

Lemma 2.3: *For a.e. $y \in \mathbb{T}$ we have*

$$\frac{1}{q_k \beta_{k-1}} L(\tilde{A}_{k,y})(x) - (-1)^k \psi(y) \to 0,$$
$$\frac{1}{q_{k-1} \beta_{k-1}} L(\tilde{C}_{k,y}^{-1})(x) - (-1)^k \psi(y) \to 0,$$

uniformly for $x \in [-1, 1]$ and $\|\psi(y)\| = d(\varphi)$.

We will denote by $\Delta(\varphi)$ the set of all points $y \in \mathbb{T}$ satisfying the properties of Lemma 2.3 and such that

$$\lim_{n\to\pm\infty} \frac{1}{n} L(\varphi^{(n)})(y) = \psi(y).$$

To prove the above lemma we need the following simple fact.

Lemma 2.4: *Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers which converges to zero. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions on $\mathbb{T}$ with nonnegative real values. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded and $f_n(x) \to 0$ for a.e. $x \in \mathbb{T}$. Then*

$$\frac{1}{c_n} \int_{y-c_n}^{y+c_n} f_n(x) dx \to 0$$

for a.e. $y \in \mathbb{T}$.

Proof: Fix $\varepsilon > 0$. By the Egoroff theorem, there exists a closed set B_ε such that $\lambda(B_\varepsilon) > 1 - \varepsilon$ and

$$\lim_{n\to\infty} \sup_{x\in B_\varepsilon} f_n(x) = 0.$$

Denote by A_ε the set of all density points of B_ε, i.e.,

$$A_\varepsilon = \left\{x \in \mathbb{T};\ \lim_{z\to 0^+} \frac{\lambda(B_\varepsilon \cap [x-z, x+z])}{2z} = 1\right\}.$$

Then $\lambda(A_\varepsilon) = \lambda(B_\varepsilon) > 1 - \varepsilon$. Assume that $y \in A_\varepsilon$. Then

$$\begin{aligned}\frac{1}{c_n}\int_{y-c_n}^{y+c_n} f_n(x)dx =& \frac{1}{c_n}\int_{[y-c_n,y+c_n]\cap B_\varepsilon} f_n(x)dx + \frac{1}{c_n}\int_{[y-c_n,y+c_n]\setminus B_\varepsilon} f_n(x)dx\\ \le& 2\sup_{x\in B_\varepsilon} f_n(x) + \frac{\lambda([y-c_n,y+c_n]\setminus B_\varepsilon)}{c_n}\sup_{x\in\mathbb{T}} f_n(x).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\frac{1}{c_n}\int_{y-c_n}^{y+c_n} f_n(x)dx \to 0$$

for every $y \in A_\varepsilon$. Consequently, letting $\varepsilon \to 0$ completes the proof. ∎

Proof of Lemma 2.3: Let us denote by $\Delta'(\varphi)$ the set of all points $y \in \mathbb{T}$ such that

$$\lim_{k\to\infty} \frac{1}{\beta_{k-1}q_k^2}\int_{y-\beta_{k-1}}^{y+\beta_{k-1}} \|DL(\varphi^{((-1)^k q_k)})(x)\|dx = 0,$$
$$\lim_{k\to\infty} \frac{1}{\beta_{k-1}q_{k-1}^2}\int_{y-2\beta_{k-1}}^{y+2\beta_{k-1}} \|DL(\varphi^{((-1)^k q_{k-1})})(x)\|dx = 0,$$
$$\lim_{n\to\pm\infty} \frac{1}{n}L(\varphi^{(n)}) = \psi(y)$$

and $\|\psi(y)\| = d(\varphi)$. By Theorems 1.1, 1.3 and Lemma 2.4, the set $\Delta'(\varphi)$ has full Lebesgue measure. We claim that $\Delta'(\varphi) \subset \Delta(\varphi)$. Assume that $y \in \Delta'(\varphi)$ and

$x \in [-1,1]$. Then

$$\begin{aligned}
&\left\|\frac{1}{\beta_{k-1}q_k}L(\tilde{A}_{k,y})(x) - \frac{1}{q_k}L(\varphi^{((-1)^k q_k)})(y)\right\| \\
&\qquad = \left\|\frac{1}{q_k}L(\varphi_y^{((-1)^k q_k)})(\beta_{k-1}x) - \frac{1}{q_k}L(\varphi_y^{((-1)^k q_k)})(0)\right\| \\
&\qquad \le \frac{1}{q_k}\int_{-\beta_{k-1}}^{\beta_{k-1}} \|DL(\varphi_y^{((-1)^k q_k)})(z)\|dz \\
&\qquad = \beta_{k-1}q_k\frac{1}{\beta_{k-1}q_k^2}\int_{y-\beta_{k-1}}^{y+\beta_{k-1}} \|DL(\varphi^{((-1)^k q_k)})(z)\|dz
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\frac{1}{\beta_{k-1}q_{k-1}}L(\tilde{C}_{k,y}^{-1})(x) - \frac{1}{q_{k-1}}L(\varphi^{((-1)^k q_{k-1})})(y)\right\| \\
&= \left\|\frac{1}{q_{k-1}}L((\varphi_y^{((-1)^{k-1} q_{k-1})})^{-1})(\beta_{k-1}x)) - \frac{1}{q_{k-1}}L(\varphi_y^{((-1)^k q_{k-1})})(0)\right\| \\
&= \left\|\frac{1}{q_{k-1}}L(\varphi_y^{((-1)^k q_{k-1})})(\beta_{k-1}(x-(-1)^k)) - \frac{1}{q_{k-1}}L(\varphi_y^{((-1)^k q_{k-1})})(0)\right\| \\
&\le \frac{1}{q_{k-1}}\int_{-2\beta_{k-1}}^{2\beta_{k-1}} \|DL(\varphi_y^{((-1)^k q_{k-1})})(z)\|dz \\
&= \beta_{k-1}q_{k-1}\frac{1}{\beta_{k-1}q_{k-1}^2}\int_{y-2\beta_{k-1}}^{y+2\beta_{k-1}} \|DL(\varphi^{((-1)^k q_{k-1})})(z)\|dz.
\end{aligned}$$

From $\beta_{k-1}q_{k-1} < \beta_{k-1}q_k < 1$, using Theorem 1.3 and Lemma 2.4 we have

$$\frac{1}{q_k\beta_{k-1}}L(\tilde{A}_{k,y})(x) - \frac{1}{q_k}L(\varphi^{((-1)^k q_k)})(y) \to 0,$$
$$\frac{1}{q_{k-1}\beta_{k-1}}L(\tilde{C}_{k,y}^{-1})(x) - \frac{1}{q_{k-1}}L(\varphi^{((-1)^k q_{k-1})})(y) \to 0,$$

uniformly for $x \in [-1,1]$ and $\|\psi(y)\| = d(\varphi)$. Moreover,

$$\frac{1}{q_{k-i}}L(\varphi^{((-1)^k q_{k-i})})(y) - (-1)^k\psi(y) \to 0$$

for $i = 0, 1$. It follows that $y \in \Delta(\varphi)$, and the proof is complete. ∎

COROLLARY 2.5: *Let $\varphi\colon \mathbb{T} \to SU(2)$ be a C^2-cocycle. Then $J(\varphi) = d(\varphi)$.*

Proof: Choose $y \in \Delta(\varphi)$. Then

$$\frac{1}{q_k\beta_{k-1}}\|L(\tilde{A}_{k,y})(x)\| \to d(\varphi),$$
$$\frac{1}{q_{k-1}\beta_{k-1}}\|L(\tilde{C}_{k,y})(x)\| \to d(\varphi),$$

uniformly for $x \in [-1,1]$. Therefore

$$\begin{aligned}J_k(y) =& q_k\beta_{k-1}\int_0^1 \frac{1}{q_k\beta_{k-1}}\|L(\tilde{A}_{k,y})(x)\|dx\\ &+ q_{k-1}\beta_k\frac{1}{\alpha_k}\int_0^{\alpha_k}\frac{1}{q_{k-1}\beta_{k-1}}\|L(\tilde{C}_{k,y})(x)\|dx\end{aligned}$$

tends to $d(\varphi)$, by (4). It follows that $J(y) = d(\varphi)$ for a.e. $y \in \mathbb{T}$, which completes the proof. ∎

LEMMA 2.6: *Let $\varphi\colon \mathbb{T} \to SU(2)$ be a C^2-cocycle. Assume that $0 \in \Delta(\varphi)$. Then*

$$\tilde{A}_k(x) - e^{L(\tilde{A}_k)(0)x}\tilde{A}_k(0) \to 0, \tag{7}$$

$$\tilde{C}_k^{-1}(x) - e^{L(\tilde{C}_k^{-1})(0)x}\tilde{C}_k^{-1}(0) \to 0, \tag{8}$$

uniformly for $x \in [-1,1]$. Moreover, the matrices $\tilde{A}_k(0)$, $\tilde{C}_k^{-1}(0)$, $L(\tilde{A}_k)(0)$ and $L(\tilde{C}_k^{-1})(0)$ asymptotically commute each other, i.e.,

$$[L(\tilde{A}_k)(0), L(\tilde{C}_k^{-1})(0)] \to 0, \tag{9}$$

$$L(\tilde{A}_k)(0) - \mathrm{Ad}_{\tilde{A}_k(0)}L(\tilde{A}_k)(0) \to 0, \tag{10}$$

$$L(\tilde{C}_k^{-1})(0) - \mathrm{Ad}_{\tilde{C}_k^{-1}(0)}L(\tilde{C}_k^{-1})(0) \to 0, \tag{11}$$

$$L(\tilde{C}_k^{-1})(0) - \mathrm{Ad}_{\tilde{A}_k(0)}L(\tilde{C}_k^{-1})(0) \to 0, \tag{12}$$

$$L(\tilde{A}_k)(0) - \mathrm{Ad}_{\tilde{C}_k^{-1}(0)}L(\tilde{A}_k)(0) \to 0, \tag{13}$$

and if $d(\varphi) \neq 0$, then

$$\tilde{A}_k(0)\tilde{C}_k^{-1}(0) - \tilde{C}_k^{-1}(0)\tilde{A}_k(0) \to 0. \tag{14}$$

Proof: First note that

$$\begin{aligned}L(\tilde{A}_k)(x) - L(e^{L(\tilde{A}_k)(0)(\cdot)}\tilde{A}_k(0))(x) &= L(\tilde{A}_k)(x) - L(\tilde{A}_k)(0) \to 0,\\ L(\tilde{C}_k^{-1})(x) - L(e^{L(\tilde{C}_k^{-1})(0)(\cdot)}\tilde{C}_k^{-1}(0))(x) &= L(\tilde{C}_k^{-1})(x) - L(\tilde{C}_k^{-1})(0) \to 0,\end{aligned}$$

uniformly for $x \in [-1,1]$ and

$$\tilde{A}_k(0) = e^{L(\tilde{A}_k)(0)0}\tilde{A}_k(0), \quad \tilde{C}_k^{-1}(0) = e^{L(\tilde{C}_k^{-1})(0)0}\tilde{C}_k^{-1}(0).$$

This implies (7) and (8). (9) follows immediately from assumption. Since

$$\frac{1}{n}(L(\varphi^{(n)})(0) - \mathrm{Ad}_{\varphi^{(n)}(0)}\, L(\varphi^{(n)})(n\alpha)) = 2\Big(\frac{1}{n}L(\varphi^{(n)})(0) - \frac{1}{2n}L(\varphi^{(2n)})(0)\Big) \to 0,$$

as $n \to \pm\infty$, we have

$$\frac{1}{q_k\beta_{k-1}}(L(\tilde{A}_k)(0) - \mathrm{Ad}_{\tilde{A}_k(0)} L(\tilde{A}_k)((-1)^k\alpha_k))$$
$$= \frac{1}{q_k}(L(\varphi^{((-1)^k q_k)})(0) - \mathrm{Ad}_{\varphi^{((-1)^k q_k)}(0)} L(\varphi^{((-1)^k q_k)})((-1)^k\beta_k)) \to 0.$$

Since $(1/q_k\beta_{k-1})(L(\tilde{A}_k)((-1)^k\alpha_k)) - L(\tilde{A}_k)(0))$ tends to zero,

$$\frac{1}{q_k\beta_{k-1}}(L(\tilde{A}_k)(0) - \mathrm{Ad}_{\tilde{A}_k(0)} L(\tilde{A}_k)(0)) \to 0.$$

Similarly,

$$\frac{1}{q_{k-1}\beta_{k-1}}(L(\tilde{C}_k^{-1})(0) - \mathrm{Ad}_{\tilde{C}_k^{-1}(0)} L(\tilde{C}_k^{-1})(0)) \to 0.$$

This leads to (10), (11), (12) and (13).

Suppose that $d(\varphi) \neq 0$. Then the sequence $\{\|L(\tilde{A}_k)(0)\|\}_{k=1}^{\infty}$ is bounded and separated from zero. Since $L(\tilde{A}_k)(0)$ asymptotically commutes with $\tilde{A}_k(0)$ and $\tilde{C}_k^{-1}(0)$, it follows that

$$\tilde{A}_k(0)\tilde{C}_k^{-1}(0) - \tilde{C}_k^{-1}(0)\tilde{A}_k(0) \to 0. \quad ∎$$

THEOREM 2.7: *Let $\varphi\colon \mathbb{T} \to SU(2)$ be a C^2-cocycle. Then $d(\varphi) \in 2\pi\mathbb{N}$.*

Proof: First note we can assume that $0 \in \Delta(\varphi)$, because degree is invariant under the rotation by any element from the circle. Then $d(\varphi) = \|\psi(0)\|$. Next assume that $d(\varphi) \neq 0$. Since $((1, \tilde{C}_k), (\alpha_k, \tilde{A}_k))$ is a $\mathbb{Z}^2$-action,

$$\tilde{A}_k(x) \cdot \tilde{C}_k(x+\alpha_k) = \tilde{C}_k(x) \cdot \tilde{A}_k(x+1) \quad \text{for any real } x.$$

Hence

$$\tilde{C}_k^{-1}(0) \cdot \tilde{A}_k(0) = \tilde{A}_k(1) \cdot \tilde{C}_k^{-1}(\alpha_k).$$

From (7) and (8),

$$\tilde{A}_k(1) - e^{L(\tilde{A}_k)(0)}\tilde{A}_k(0) \to 0,$$
$$\tilde{C}_k^{-1}(\alpha_k) - e^{L(\tilde{C}_k^{-1})(0)\alpha_k}\tilde{C}_k^{-1}(0) \to 0.$$

Therefore

$$\tilde{C}_k^{-1}(0)\tilde{A}_k(0) - e^{L(\tilde{A}_k)(0)}\tilde{A}_k(0)e^{L(\tilde{C}_k^{-1})(0)\alpha_k}\tilde{C}_k^{-1}(0) \to 0.$$

Applying (9)–(14) we get

$$e^{L(\tilde{A}_k)(0)+L(\tilde{C}_k^{-1})(0)\alpha_k} \to \mathrm{Id}.$$

On the other hand,

$$L(\tilde{A}_k)(0) + L(\tilde{C}_k^{-1})(0)\alpha_k$$
$$= q_k\beta_{k-1}\frac{1}{q_k\beta_{k-1}}L(\tilde{A}_k)(0) + q_{k-1}\beta_k\frac{1}{q_{k-1}\beta_{k-1}}L(\tilde{C}_k^{-1})(0) \to \psi(0).$$

Therefore $e^{\psi(0)}=\mathrm{Id}$ and $d(\varphi) = \|\psi(0)\| = 2\pi r$, where $r \in \mathbb{N}$. ∎

By the same method as in the proof of Theorem 6.3 of [6] one can prove the following result.

LEMMA 2.8: *Let $\varphi\colon \mathbb{T} \to SU(2)$ be a C^∞-cocycle and let N be an infinite subset of $\mathbb{N}$. Suppose that φ satisfies (7)–(14) and*

$$\|L(\tilde{A}_k)(0) + L(\tilde{C}_k^{-1})(0)\alpha_k\| \to 2\pi r,$$

where $r \in \mathbb{N}\setminus\{0\}$. Then there exist an increasing sequence $\{n_k\}_{k=1}^\infty$ in N, a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{T}, SU(2))$ and a real number w such that $((1,\tilde{C}_k),(\alpha_{n_k},\tilde{A}_{n_k}))$ and $((1,\mathrm{Id}),(\alpha_{n_k},\varphi_k))$ are C^∞-cohomologous and

$$\lim_{k\to\infty}\|\varphi_k - \exp_{r,w}\|_{C^s} = 0$$

for any natural s.

Additionally, applying Lemmas 2.3–2.8 and Lemmas 2.1 and 2.2 gives the following conclusion.

COROLLARY 2.9: *Let $\varphi\colon \mathbb{T} \to SU(2)$ be a C^∞-cocycle with $d(\varphi) = 2\pi r \neq 0$ and let N be an infinite subset of $\mathbb{N}$. Then there exist $y \in \mathbb{T}$, an increasing sequence $\{n_k\}_{k=1}^\infty$ in N, a sequence $\{\varphi_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{T}, SU(2))$ and $w \in \mathbb{R}$ such that*

- *the $\mathbb{Z}^2$-actions $\widetilde{R}_{n_k}((1,\mathrm{Id}),(\alpha,\varphi_y))$ and $((1,\mathrm{Id}),(\alpha_{n_k},\varphi_k))$ are C^∞-cohomologous,*
- *$d(\varphi_k) = d(\varphi) = 2\pi r$,*
- *$\lim_{k\to\infty}\|\varphi_k - \exp_{r,w}\|_{C^s} = 0$ for any natural s.*

2.4 MEASURABLE INVARIANCE OF DEGREE. It is easy to see that degree is invariant under C^1-cohomology. In the simplest case $G = \mathbb{T}$, degree is invariant even under measurable cohomology, but the proof of this fact does not work in the nonabelian case. Nevertheless, applying the renormalization algorithm we are able to show measurable invariance of degree for C^2-cocycles.

THEOREM 2.10: *Suppose that two C^2-cocycles $\varphi_1, \varphi_2\colon \mathbb{T} \to SU(2)$ are measurably cohomologous. Then $d(\varphi_1) = d(\varphi_2)$.*

Proof: Let $B\colon \mathbb{T} \to SU(2)$ be a measurable transfer function, i.e.,

$$B(x)^{-1} \cdot \varphi_1(x) \cdot B(x+\alpha) = \varphi_2(x).$$

Let us denote by $\Delta^*(B)$ the set of all $y \in \mathbb{T}$ such that

$$\lim_{x\to 0} \frac{1}{x} \int_0^x \|B(y+\tau) - B(y)\| = 0.$$

The set $\Delta^*(B)$ has full Lebesgue measure. Suppose that $y \in \Delta^*(B)$. Next, for every natural k denote by $B_{k,y}\colon \mathbb{T} \to SU(2)$ the function $B_{k,y}(x) = B(\beta_k x + y)$. Then

$$\begin{aligned}\|B_{k,y} - B(y)\|_{L^2[0,2]} &= \int_0^2 \|B(\beta_k x + y) - B(y)\| dx\\ &= \frac{1}{\beta_k} \int_0^{2\beta_k} \|B(y+\tau) - B(y)\| d\tau \to 0.\end{aligned}$$

For simplicity of notation let us assume that $0 \in \Delta(\varphi_1) \cap \Delta(\varphi_2) \cap \Delta^*(B)$ and we will write B_k instead of $B_{k,0}$. Since

$$(0,B) \circ ((1,\mathrm{Id}),(\alpha,\varphi_1)) \circ (0,B)^{-1} = ((1,\mathrm{Id}),(\alpha,\varphi_2)),$$

we have

$$(0,B_k) \circ ((1,\tilde{C}_k(\varphi_1)),(\alpha_k,\tilde{A}_k(\varphi_1))) \circ (0,B_k)^{-1} = ((1,\tilde{C}_k(\varphi_2)),(\alpha_k,\tilde{A}_k(\varphi_2)))$$

for any natural k. It follows that

$$B_k(x)^{-1} \cdot \tilde{A}_k(\varphi_1)(x) \cdot B_k(x+\alpha_k) = \tilde{A}_k(\varphi_2)(x).$$

Next choose an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of even numbers such that

$$\tilde{A}_{n_k}(\varphi_i)(0) \to A_i \in SU(2),$$

for $i = 1,2$ and

$$q_{n_k}\beta_{n_k-1} \to a.$$

Then

$$L(\tilde{A}_{n_k})(\varphi_i)(x) \to a\,\psi(\varphi_i)(0)$$

uniformly for $x \in [-1,1]$ for $i = 1,2$ and $0 < a$, by (1). Moreover,

$$\tilde{A}_{n_k}(\varphi_i)(x) \to e^{a\psi(\varphi_i)(0)x} A_i$$

uniformly for $x \in [-1,1]$ for $i = 1,2$, by (7). Since $B_k \to B(0)$ in $L^2[0,2]$,

$$B(0)^{-1} \cdot e^{a\psi(\varphi_1)(0)x} A_1 \cdot B(0) = e^{a\psi(\varphi_2)(0)x} A_2$$

on $[0,1]$. This leads to

$$\begin{aligned} ad(\varphi_1) &= a\|\psi(\varphi_1)(0)\| = \|L(e^{a\psi(\varphi_1)(0)x} A_1)\| \\ &= \|L(B(0)^{-1} \cdot e^{a\psi(\varphi_1)(0)x} A_1 \cdot B(0))\| \\ &= \|L(e^{a\psi(\varphi_2)(0)x} A_2)\| = a\|\psi(\varphi_2)(0)\| = ad(\varphi_2). \end{aligned}$$

Since $0 < a$, we conclude that $d(\varphi_1) = d(\varphi_2)$. ∎

3. The case of rotations satisfying a Diophantine condition

For every $\gamma > 0$ and $\sigma > 1$ define

$$CD(\gamma, \sigma) = \{\alpha \in \mathbb{T}; \forall_{k \in \mathbb{N}\setminus\{0\},\, l \in \mathbb{Z}}\ |k\alpha - l| > 1/(\gamma k^\sigma)\}.$$

Let us denote by Σ the set of all $\alpha \in \mathbb{T}$ such that there exist $\gamma > 0$ and $\sigma > 1$ for which $\alpha_k \in CD(\gamma, \sigma)$ for infinitely many k. Since any set $CD(\gamma, \sigma)$ has positive Lebesgue measure, the set Σ has full Lebesgue measure, by the ergodicity of the Gauss transformation. In this section we prove the following result.

THEOREM 3.1: *Let $\alpha \in \Sigma$. Suppose that $\varphi: \mathbb{T} \to SU(2)$ is a C^∞-cocycle with $d(\varphi) = 2\pi r \neq 0$. Then φ is C^∞-cohomologous to a cocycle $\exp_{r,w}$, where w is a real number.*

To prove it we need the following fact.

LEMMA 3.2: *For every $\gamma > 0$, $\sigma > 1$ and $r \in \mathbb{N} \setminus \{0\}$ there exist $s_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for any $\alpha \in CD(\gamma, \sigma)$ and any $\varphi \in C^\infty(\mathbb{T}, SU(2))$ if*
- $\|\varphi - \exp_{r,0}\|_{C^{s_0}} < \varepsilon_0$,
- $d(\varphi) = 2\pi r \neq 0$,

then φ is C^∞-cohomologous to a cocycle $\exp_{r,w}$, where w is a real number.

The above lemma (its proof will be given later) is a conclusion from the following result proved by R. Krikorian [6, Th. 9.1].

PROPOSITION 3.3: *For every $\gamma > 0$, $\sigma > 1$ and $r \in \mathbb{N} \setminus \{0\}$ there exist $s_0 = s_0(\gamma, \sigma, r) \in \mathbb{N}$ and $\varepsilon_0 = \varepsilon_0(\gamma, \sigma, r) > 0$ such that for any $\alpha \in CD(\gamma, \sigma) \cap (1/5, 1/4)$ and any $\varphi \in C^\infty(\mathbb{T}, SU(2))$ if $\|\varphi - \exp_{r,0}\|_{C^{s_0}} < \varepsilon_0$, then*
- *either $J(\varphi) < 2\pi r$,*

- *or φ is C^∞-cohomologous to a cocycle* $\exp_{r,w}$, *where w is a real number.*

Proof of Theorem 3.1: Take $\gamma > 0$ and $\sigma > 1$ such that the set $N = \{k \in \mathbb{N};\ \alpha_k \in CD(\gamma,\sigma)\}$ is infinite. Choose $s_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ satisfying the properties of Lemma 3.2. By Corollary 2.9, there exist $y \in \mathbb{T}$, an increasing sequence $\{n_k\}_{k=1}^{\infty}$ in N, a sequence $\{\varphi_k\}_{k=1}^{\infty}$ in $C^\infty(\mathbb{T}, SU(2))$ and $w_1 \in \mathbb{R}$ such that

- $\alpha_{n_k} \in CD(\gamma,\sigma)$,
- the $\mathbb{Z}^2$-actions $\widetilde{\mathcal{R}}_{n_k}((1,\mathrm{Id}),(\alpha,\varphi_y))$ and $((1,\mathrm{Id}),(\alpha_{n_k},\varphi_k))$ are C^∞-cohomologous,
- $d(\varphi_k) = d(\varphi) = 2\pi r$,
- $\varphi_k \to \exp_{r,w_1}$ in $C^{s_0}(\mathbb{T}, SU(2))$.

Let k be a natural number such that $\|\varphi_k - \exp_{r,w_1}\|_{C^{s_0}} < \varepsilon_0$. By Lemma 3.2, φ_k is C^∞-cohomologous to a cocycle $\exp_{r,w_2}$, where w_2 is a real number. Let $A, C\colon \mathbb{T} \to \mathfrak{T}$ be C^∞-functions such that

$$\begin{aligned}(1,C) &= (\beta_{n_k-1},\mathrm{Id})^{q_{n_k}} \circ (\beta_{n_k}, \exp_{r,w_2} \circ S_{\beta^{-1}_{n_k-1}})^{q_{n_k-1}},\\ (\alpha,A) &= (\beta_{n_k-1},\mathrm{Id})^{p_{n_k}} \circ (\beta_{n_k}, \exp_{r,w_2} \circ S_{\beta^{-1}_{n_k-1}})^{p_{n_k-1}}.\end{aligned}$$

Since $\widetilde{\mathcal{R}}_{n_k}((1,\mathrm{Id}),(\alpha,\varphi_y))$ and $((1,\mathrm{Id}),(\alpha_{n_k},\exp_{r,w_2}))$ are C^∞-cohomologous, $\mathcal{R}_{n_k}((1,\mathrm{Id}),(\alpha,\varphi_y))$ and $((\beta_{n_k-1},\mathrm{Id}),(\beta_{n_k},\exp_{r,w_2} \circ S_{\beta^{-1}_{n_k-1}}))$ are C^∞-cohomologous, too. From (5) we see that $((1,\mathrm{Id}),(\alpha,\varphi_y))$ and $((1,C),(\alpha,A))$ are C^∞-cohomologous. Moreover, $((1,C),(\alpha,A))$ is C^∞-cohomologous to a $\mathbb{Z}^2$-action of the form $((1,\mathrm{Id}),(\alpha,\xi))$, where $\xi\colon \mathbb{T} \to \mathfrak{T}$ is a C^∞-cocycle. Then the cocycles φ_y and ξ are C^∞-cohomologous. Let $g\colon \mathbb{T} \to \mathbb{T}$ be a C^∞-cocycle such that

$$\xi(x) = \begin{bmatrix} g(x) & 0 \\ 0 & \overline{g(x)} \end{bmatrix}$$

and $d(g) \geq 0$. If $d(g) \leq 0$, then we can take

$$\xi(x) = \begin{bmatrix} \overline{g(x)} & 0 \\ 0 & g(x) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} g(x) & 0 \\ 0 & \overline{g(x)} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1},$$

which is also C^∞-cohomologous to φ_y. It follows that $d(g) = r$. Since α is Diophantine, g is C^∞-cohomologous to a cocycle of the form $\mathbb{T} \ni x \mapsto e^{2\pi i(rx+w_3)} \in \mathbb{T}$, where w_3 is a real number. It follows that φ is C^∞-cohomologous to the cocycle $\exp_{r,w_3-y}$, which completes the proof. ∎

3.1 Lacking proof. To prove Lemma 3.2 we need the following facts.

LEMMA 3.4: *For every $\gamma > 0$ and $\sigma > 1$ there exist $\gamma' > 0$, $\sigma' > 1$ and $M \in \mathbb{N}$ such that for every $\alpha \in CD(\gamma,\sigma)$ there exists a natural number $1 \le m \le M$ for which $m\alpha \in (1/5, 1/4) \cap CD(\gamma',\sigma')$.*

Proof: First recall that if $\alpha \in CD(\gamma,\sigma)$, then

$$\frac{1}{\gamma q_n^{\sigma-1}} < q_n|q_n\alpha - p_n| < \frac{1}{q_{n+1}}.$$

Hence $q_{n+1} < \gamma q_n^{\sigma-1}$ for any natural n. It follows that there exists $C = C_{\gamma,\sigma} > 0$ such that $\alpha \in CD(\gamma,\sigma)$ implies $20 < q_7 < C$.

Suppose that $\alpha \in CD(\gamma,\sigma)$. Then

$$\frac{1}{2C} < \frac{1}{2q_7} < \{q_6\alpha\} < \frac{1}{q_7} < \frac{1}{20}.$$

It follows that there exists a natural number $1 \le m' < C$ such that

$$\frac{1}{5} < \{m'q_6\alpha\} < \frac{1}{4}.$$

Hence there exists a natural number $1 \le m \le M = C^2$ such that $m\alpha \in (1/5, 1/4)$. Moreover,

$$|km\alpha - l| > \frac{1}{\gamma m^\sigma k^\sigma} \ge \frac{1}{\gamma M^\sigma k^\sigma}$$

for any $k \in \mathbb{N} \setminus \{0\}$ and $l \in \mathbb{Z}$. Therefore we can take $M = C^2$, $\gamma' = \gamma M^\sigma$ and $\sigma' = \sigma$, and the proof is complete. ∎

LEMMA 3.5: *Let $\varphi: \mathbb{T} \to SU(2)$ and $\xi: \mathbb{T} \to \mathfrak{T}$ be C^∞ cocycles. Let $m \neq 0$ be a natural number. Suppose that $\varphi^{(m)}$ and $\xi^{(m)}$ are C^∞-cohomologous as cocycles over the rotation T^m and $d(\varphi) \neq 0$. Then there exists $A \in \mathfrak{T}$ such that the cocycles φ and $\xi \cdot A$ are C^∞-cohomologous, too.*

Proof: Let $g: \mathbb{T} \to \mathbb{T}$ be a C^∞-cocycle such that

$$\xi(x) = \begin{bmatrix} g(x) & 0 \\ 0 & \overline{g(x)} \end{bmatrix}.$$

Then $2\pi|d(g)| = d(\xi) = d(\varphi) \neq 0$. By Theorem 1.2, there exist measurable functions $p: \mathbb{T} \to SU(2)$ and $\gamma: \mathbb{T} \to \mathbb{T}$ such that

$$\varphi(x) = p(x) \begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix} p(Tx)^{-1}. \tag{15}$$

Let $q\colon \mathbb{T} \to SU(2)$ be a C^∞-function such that

$$\varphi^{(m)}(x) = q(x)\xi^{(m)}(x)q(T^m x)^{-1}.$$

Then

$$p(x)\begin{bmatrix} \gamma^{(m)}(x) & 0 \\ 0 & \overline{\gamma^{(m)}(x)} \end{bmatrix} p(T^m x)^{-1} = q(x)\begin{bmatrix} g^{(m)}(x) & 0 \\ 0 & \overline{g^{(m)}(x)} \end{bmatrix} q(T^m x)^{-1}.$$

Let

$$q^{-1}p = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},$$

where $a, b\colon \mathbb{T} \to \mathbb{C}$ are measurable functions such that $|a|^2 + |b|^2 = 1$. Then

$$\begin{aligned} a \cdot \gamma^{(m)} =& g^{(m)} \cdot (a \circ T^m), \\ b \cdot \overline{\gamma^{(m)}} =& g^{(m)} \cdot (b \circ T^m), \\ ab =& (g^{(m)})^2 \cdot ((ab) \circ T^m). \end{aligned}$$

Since $d((g^{(m)})^2) = 2md(g) \neq 0$, we get either $a \equiv 0$ or $b \equiv 0$.

Case 1: Suppose that $b \equiv 0$. Then $a\colon \mathbb{T} \to \mathbb{T}$ and $\gamma_{\bar{a}}^{(m)} = g^{(m)}$. Moreover,

$$\varphi(x) = q(x)\begin{bmatrix} \gamma_{\bar{a}}(x) & 0 \\ 0 & \overline{\gamma_{\bar{a}}(x)} \end{bmatrix} q(Tx)^{-1},$$

by (15). Hence $\gamma_{\bar{a}}\colon \mathbb{T} \to \mathbb{T}$ is a C^∞-cocycle. Using a standard Fourier analysis method we can assert that there exists $l \in \mathbb{N}$ such that $\gamma_{\bar{a}} = g \cdot e^{2\pi i l/m}$. Therefore

$$\varphi(x) = q(x)\xi(x)\begin{bmatrix} e^{2\pi i l/m} & 0 \\ 0 & e^{-2\pi i l/m} \end{bmatrix} q(Tx)^{-1}.$$

Case 2: Suppose that $a \equiv 0$. Then $b\colon \mathbb{T} \to \mathbb{T}$ and $\overline{\gamma_b^{(m)}} = g^{(m)}$. Moreover,

$$\begin{aligned} \varphi(x) =& q(x)\begin{bmatrix} 0 & b(x) \\ -\overline{b(x)} & 0 \end{bmatrix}\begin{bmatrix} \gamma(x) & 0 \\ 0 & \overline{\gamma(x)} \end{bmatrix}\begin{bmatrix} 0 & -b(Tx) \\ \overline{b(Tx)} & 0 \end{bmatrix} q(Tx)^{-1} \\ =& q(x)\begin{bmatrix} \overline{\gamma_b(x)} & 0 \\ 0 & \gamma_b(x) \end{bmatrix} q(Tx)^{-1}, \end{aligned}$$

by (15). Hence $\gamma_b\colon \mathbb{T} \to \mathbb{T}$ is a C^∞-cocycle and there exists $l \in \mathbb{N}$ such that $\overline{\gamma_b} = g \cdot e^{2\pi i l/m}$, which completes the proof. ∎

Proof of Lemma 3.2: Fix $\gamma > 0$, $\sigma > 1$ and $r \in \mathbb{N} \setminus \{0\}$. Let $\gamma' > 0$, $\sigma' > 1$ and $M \in \mathbb{N}$ be constants satisfying the properties of Lemma 3.4. Take

$s_0 = s_0(\gamma', \sigma') \in \mathbb{N}$ and $\varepsilon' = \varepsilon_0(\gamma', \sigma') \in (0, 1)$ satisfying the properties of Proposition 3.3. Next choose $K, R > 0$ such that

$$\|\varphi^{(m)} - \psi^{(m)}\|_{C^{s_0}} \leq K\|\varphi - \psi\|_{C^{s_0}}(1 + \|\varphi\|_{C^{s_0}})^R(1 + \|\psi\|_{C^{s_0}})^R$$

for any irrational α, any cocycles $\varphi, \psi \in C^\infty(\mathbb{T}, SU(2))$ and any natural $1 \leq m \leq M$. Define $\varepsilon_0 = \varepsilon'/(K(2\pi r + 2)^{2R})$.

Suppose that $\alpha \in CD(\gamma, \sigma)$ and φ is a C^∞-cocycle such that

$$\|\varphi - \exp_{r,0}\|_{C^{s_0}} < \varepsilon_0 \quad \text{and} \quad d(\varphi) = 2\pi r.$$

Then there exist a natural number $1 \leq m \leq M$ such that

$$m\alpha \in CD(\gamma', \sigma') \cap (1/5, 1/4).$$

Therefore

$$\begin{aligned}\|\varphi^{(m)} - \exp_{r,0}^{(m)}\|_{C^{s_0}} &\leq K\|\varphi - \exp_{r,0}\|_{C^{s_0}}(1 + \|\varphi\|_{C^{s_0}})^R(1 + \|\exp_{r,0}\|_{C^{s_0}})^R \\ &< K\varepsilon_0(\|\exp_{r,0}\|_{C^{s_0}} + 2)^{2R} \\ &< K\varepsilon_0(2\pi r + 2)^{2R} = \varepsilon'.\end{aligned}$$

Moreover, $J(\varphi^{(m)}) = d(\varphi^{(m)}) = 2\pi rm$ and $\exp_{r,0}^{(m)} = \exp_{rm,v}$, where $v = rm(m-1)\alpha/2$. By Proposition 3.3, $\varphi^{(m)}$ is C^∞-cohomologous to a cocycle $\exp_{r,w'}^{(m)}$. Applying Lemma 3.5 we conclude that φ is C^∞-cohomologous to a cocycle $\exp_{r,w}$, where w is a real number. ∎

A. More about degree

One may ask whether the degree of a cocycle depends on the base rotation or only on the function, which creates the cocycle. Of course, the degree of a cocycle is independent of the base rotation in the case where $G = \mathbb{T}$. A different phenomenon occurs in the case where $G = SU(2)$. For any irrational $\alpha \in \mathbb{T}$ and any C^1-function $\varphi: \mathbb{T} \to SU(2)$ we will denote by $d(\varphi, \alpha)$ the degree of the cocycle φ over the rotation by α. In this section we show that for any two distinct $\alpha_1, \alpha_2 \in \mathbb{T}$ with $\alpha_1 - \alpha_2 \neq 1/2$ there exists a C^∞-function $\varphi: \mathbb{T} \to SU(2)$ for which $d(\varphi, \alpha_1) \neq d(\varphi, \alpha_2)$. For every $\beta \in \mathbb{T}$ let $\rho_\beta: \mathbb{T} \to SU(2)$ be given by

$$\rho_\beta(x) = \begin{bmatrix} e^{2\pi ix} & 0 \\ 0 & e^{-2\pi ix} \end{bmatrix} \begin{bmatrix} \cos 2\pi\beta & \sin 2\pi\beta \\ -\sin 2\pi\beta & \cos 2\pi\beta \end{bmatrix}.$$

To construct the desired function, we have to know $d(\rho_\beta, \alpha)$ for any irrational α. Obviously, if β is equal to 0 or 1/2, then $d(\rho_\beta, \alpha) = 2\pi$ for any irrational α.

Suppose that $0 \neq \beta \neq 1/2$. It is easy to check that $\|L(\rho_\beta^{(2)})(x)\| = 4\pi|\cos 2\pi\beta|$ for any $x \in \mathbb{T}$ and any irrational α. Therefore

$$d(\rho_\beta, \alpha) = \inf_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{n}\|L(\rho_\beta^{(n)})\|_{L^1(\mathbb{T})} \leq \frac{1}{2}\|L(\rho_\beta^{(2)})\|_{L^1(\mathbb{T})} = 2\pi|\cos 2\pi\beta| < 2\pi.$$

It follows that $d(\rho_\beta, \alpha) = 0$ for any irrational α.

THEOREM A.1: *Let α_1, α_2 be distinct elements of $\mathbb{T}$ such that $\alpha_1 - \alpha_2 \neq 1/2$. Then there exists a C^∞-function $\varphi\colon \mathbb{T} \to SU(2)$ for which $d(\varphi, \alpha_1) \neq d(\varphi, \alpha_2)$.*

Proof: Set

$$A = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Then

$$\begin{bmatrix} \cos 2\pi\beta & \sin 2\pi\beta \\ -\sin 2\pi\beta & \cos 2\pi\beta \end{bmatrix} = \mathrm{Ad}_A \begin{bmatrix} e^{2\pi i\beta} & 0 \\ 0 & e^{-2\pi i\beta} \end{bmatrix}$$

for any $\beta \in \mathbb{T}$. Define

$$\varphi(x) = \begin{bmatrix} e^{2\pi ix} & 0 \\ 0 & e^{-2\pi ix} \end{bmatrix} A^{-1} \begin{bmatrix} e^{2\pi ix} & 0 \\ 0 & e^{-2\pi ix} \end{bmatrix} A \begin{bmatrix} e^{-2\pi i(x+\alpha_1)} & 0 \\ 0 & e^{2\pi i(x+\alpha_1)} \end{bmatrix}.$$

Then

$$\varphi(x) = \begin{bmatrix} e^{2\pi ix} & 0 \\ 0 & e^{-2\pi ix} \end{bmatrix} A^{-1}\rho_{\alpha_2-\alpha_1}(x)A \begin{bmatrix} e^{-2\pi i(x+\alpha_2)} & 0 \\ 0 & e^{2\pi i(x+\alpha_2)} \end{bmatrix}.$$

Therefore φ and ρ_0 are C^∞-cohomologous as cocycle over the rotation by α_1 and φ and $\rho_{\alpha_2-\alpha_1}$ are C^∞-cohomologous as cocycle over the rotation by α_2. It follows that

$$\begin{aligned} d(\varphi, \alpha_1) &= d(\rho_0, \alpha_1) = 1, \\ d(\varphi, \alpha_2) &= d(\rho_{\alpha_2-\alpha_1}, \alpha_2) = 0, \end{aligned}$$

and the proof is complete. ∎

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